Studies of resonance conditions on neutrino oscillations in matter

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Abstract

We analytically discuss the resonance conditions among several neutrinos in matter. The discriminant for the characteristic equation of the Hamiltonian is expressed by the coefficients of the equation. The result of the computation for the discriminants tells us that the neutrino energy and the matter density are in inverse proportion to each other at the resonance states in not only 2- but also 3- and 4-neutrino models.

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I. INTRODUCTION

Neutrino oscillations mean transitions among neutrino flavors, and give an interpretation of neutrino phenomena[1, 2, 3, 4] found in the observations of the solar and the atmospheric neutrinos and the LSND experiment. The mass squared differences are some parameters of the neutrino oscillations, and are determined by some results of the observations and the experiments mentioned above. In order to explain the results of those within one framework, the three kinds of mass squared differences are needed. Existence of them means that four neutrinos have different masses from each other. It is a reason to discuss the 4-neutrino oscillation. Four neutrinos are classified into active neutrino flavors $(\nu_e, \nu_\mu, \nu_\tau)$, which interact with leptons in the weak interaction, and sterile neutrino (ν_s). The sterile neutrino does not have the weak interaction.

Recent analyses of the experiments and the observations disfavor the four neutrino flavors. The possibility of an oscillation $\nu_e \leftrightarrow \nu_s$ is strongly excluded by the analyses. However, the result of the LSND experiment causes the maximum one of the three mass squared differences, and gives grounds for the 4-neutrino models. The upcoming MiniBooNE experiments [5] may form a conclusion about this discrepancy. Whatever conclusion the experiment leads, it is useful to consider the neutrino oscillations with the sterile neutrino in the condition different from that of the experiment.

The neutrino oscillation pattern in vacuum can get modified when the neutrinos pass through matter. This is known as the Mikheyev-Smirnov-Wolfenstein (MSW) effects[6], which can be described by an effective Hamiltonian. The Hamiltonian is expressed by the sum of the vacuum Hamiltonian and the interaction with the charged and the neutral currents[7]. In order to write the 4×4 Hamiltonian matrix in matter, we introduce the 4×4 mixing matrix which is an extension of the 3×3 Maki-Nakagawa-Sakata-Pontecorvo (MNSP) matrix[8, 9].

Analytical calculations of active 3-neutrino oscillations in matter have been performed[10, 11, 12, 13, 14]. In our previous paper[11], we have given analytic expressions for the time-evolution operator in the 4-neutrino oscillation and the transition probabilities in the presence of constant matter densities.

In the results of our calculations, the value of the matter density which causes the neutrino resonance varies with the average of the neutrino energies. In general, the resonance occurs

when two of the energy levels approach to each other. However, the calculation to get the energy levels is difficult because we have to solve the characteristic equation of the Hamiltonian. Then we propose to use the discriminant of the characteristic equation made from the Hamiltonian in matter in order to estimate the neutrino resonance conditions. In this calculation, we need not solve the characteristic equation.

The outline of the article is as follows. In Sec. II, two kinds of bases to express neutrino states are introduced. These bases are connected to each other by a mixing matrix. To describe neutrino oscillations in matter, the effective Hamiltonian with charged and neutral currents is given. In Sec. III, we discuss the resonance conditions of neutrinos due to the characteristic equation of the effective Hamiltonian in matter. We then make use of a definition of the discriminant for an algebraic equation and a relation of the coefficients and solutions for the equations. In Sec. IV, the discriminant is concretely computed within a range expected that the neutrino mixings occur. And the graphical expressions of the discriminant are illustrated. Finally, we discuss the resonance conditions for the neutrino mixing in matter in Sec. V.

II. FORMALISM

We analyze the matter effects of 4-neutrino oscillation[11], similar to those of 2- and 3-neutrino oscillations.

A. Two bases and a mixing matrix

Neutrinos are produced in flavor eigenstates $|\nu_\alpha\rangle (\alpha = e, \mu, \tau, s)$. Between the source and the detector, the neutrinos evolve as mass eigenstates $|\nu_a\rangle (a = 1, 2, 3, 4)$. There are two kinds of eigenstates: $|\nu_\alpha\rangle$ and $|\nu_a\rangle$. These eigenstates are defined by neutrino fields ν_α and ν_a corresponding to each eigenstate: $\nu^\dagger|0\rangle \equiv |\nu\rangle$, $|\nu_\alpha\rangle \equiv |\alpha\rangle$, $|\nu_a\rangle \equiv |a\rangle$, where a vacuum state is given by $|0\rangle$. In the present analysis, we will use the plane wave approximation of the fields. In this approximation, a neutrino flavor field ν_α is expressed by a linear combination of neutrino mass field ν_a:

$$\nu_\alpha = \sum_{a=1}^{4} U_{\alpha a} \nu_a, \tag{1}$$

where U is a 4×4 unitary matrix with the elements $U_{\alpha a}$. If we write this relation in neutrino eigenstates, then

$$|\alpha\rangle = \sum_{a=1}^{4} U_{\alpha a}^* |a\rangle. \tag{2}$$

An arbitrary neutrino state ψ is expressed in both the flavor and the mass bases as

$$\psi \equiv \sum_{\alpha=e,\mu,\tau,s} \psi_\alpha |\alpha\rangle = \sum_{a=1}^{4} \psi_a |a\rangle, \tag{3}$$

where ψ_α and ψ_a are the components of ψ of the flavor eigenstate basis and the mass eigenstate basis, respectively. They are related to each other in the form

$$\psi_a = \sum_{\alpha=e,\mu,\tau,s} U_{\alpha a}^* \psi_\alpha. \tag{4}$$

If we define the matrix elements as

$$\psi^{\text{(flavor)}} = (\psi_\alpha) = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \\ \psi_s \end{pmatrix}, \quad \psi^{\text{(mass)}} = (\psi_a) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \tag{5}$$

$$U = (U_{\alpha a}) = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} & U_{e4} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} & U_{\mu 4} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & U_{\tau 4} \\ U_{s1} & U_{s2} & U_{s3} & U_{s4} \end{pmatrix}, \tag{6}$$

the relations (4) of the components between the flavor and the mass eigenstates are rewritten in the matrix form,

$$\psi^{\text{(flavor)}} = U \psi^{\text{(mass)}}. \tag{7}$$

The unitary matrix U is the mixing matrix of four neutrinos. There are 6 mixing angles and 3 phases as parameters of U, in the case of 4 neutrinos.

A parameterization for U is given by

$$U = R_{34} \tilde{R}_{24} \tilde{R}_{14} R_{23} \tilde{R}_{13} R_{12}, \tag{8}$$

where R_{ab} and $\tilde{R}_{ab}$ are the real and the complex mixing matrices, respectively[15]. The structure of these matrices is

$$(R_{ab})_{ij} = \delta_{ij} + (\cos\theta_{ab} - 1)(\delta_{ai}\delta_{aj} + \delta_{bi}\delta_{bj})$$

$$+\sin\theta_{ab}(\delta_{ai}\delta_{bj} - \delta_{bi}\delta_{aj}), \tag{9}$$

$$(\tilde{R}_{ab})_{ij} = \delta_{ij} + (\cos\theta_{ab} - 1)(\delta_{ai}\delta_{aj} + \delta_{bi}\delta_{bj})$$

$$+\sin\theta_{ab}(e^{-i\Delta_{ab}}\delta_{ai}\delta_{bj} - e^{i\Delta_{ab}}\delta_{bi}\delta_{aj}), \tag{10}$$

where the mixing of mass eigenstates $|a\rangle$ and $|b\rangle$ is described by 6 mixing angles θ_{ab}: θ_{12}, θ_{13}, θ_{14}, θ_{23}, θ_{24}, θ_{34} and 3 phases Δ_{ab}: Δ_{13}, Δ_{24}, Δ_{34}. In the case of leaving the CP violations out of consideration, i.e., $\Delta_{ab} = 0$, $\tilde{R}_{ab} = R_{ab}$, and the mixing matrix U is a real orthogonal matrix[11].

B. Hamiltonian in matter

In the mass eigenstate basis, the Hamiltonian $\mathcal{H}_0$ participating in the propagation of neutrinos in vacuum is given by

$$\mathcal{H}_0 = \mathrm{diag}(E_1, E_2, E_3, E_4), \tag{11}$$

where E_a ($a = 1, 2, 3, 4$) are the energies of the neutrino mass eigenstates $|a\rangle$ with mass m_a:

$$E_a = \sqrt{m_a{}^2 + \boldsymbol{p}^2}. \tag{12}$$

Here and hereafter, we assume the momentum $\boldsymbol{p}$ to be the same for all mass eigenstates. If we assume $|\boldsymbol{p}| \gg m_a$,

$$E_a \sim |\boldsymbol{p}| + \frac{m_a{}^2}{2|\boldsymbol{p}|} \sim |\boldsymbol{p}| + \frac{m_a{}^2}{2E}, \tag{13}$$

where E is the average of the neutrino energies.

The interactions between neutrinos and matter are described by two kinds of additional potentials. One is the interaction of the charged particles (electrons) and its neutrino ν_e:

$$V_{\mathrm{CC}} = V_e = \sqrt{2}G_F\,\mathrm{diag}(N_e, 0, 0, 0). \tag{14}$$

The other is the interaction of the neutral particles(e.g., the neutron) and active neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$:

$$V_{\mathrm{NC}} = \sqrt{2}G_F\,\mathrm{diag}(-\frac{1}{2}N_n, -\frac{1}{2}N_n, -\frac{1}{2}N_n, 0), \tag{15}$$

where G_F, N_e, and N_n are the Fermi weak coupling constant, the electron number density and the neutral particle number density, respectively. Note that we assume the particle

number densities to be constant throughout the matter where the neutrinos are propagating. The interaction term (15) can be separated into two parts as

$$V_{\mathrm{NC}} = V_n + V',$$ (16)

$$V_n = \sqrt{2}G_F \, \mathrm{diag}(0, 0, 0, +\frac{1}{2}N_n),$$ (17)

$$V' = \sqrt{2}G_F \, \mathrm{diag}(-\frac{1}{2}N_n, -\frac{1}{2}N_n, -\frac{1}{2}N_n, -\frac{1}{2}N_n).$$ (18)

The interaction terms V_e and V_n are added to vacuum Hamiltonian $\mathcal{H}_0$ for the propagation of neutrinos in matter. However, the vacuum Hamiltonian and the interaction terms are written by the mass and the flavor eigenstates, respectively.

The mass eigenstate is convenient for describing the neutrinos propagation in matter. The Hamiltonian in the mass basis is

$$\begin{aligned}
\mathcal{H}_m^{(\mathrm{mass})} &= \mathcal{H}_0 + U^{-1}V_{\mathrm{CC}}U + U^{-1}V_{\mathrm{NC}}U \\
&= \mathcal{H}_0 + U^{-1}V_e U + U^{-1}V_n U - A_n I,
\end{aligned}$$ (19)

where I is the 4×4 unit matrix. The matter densities A_e, A_n and A are defined by

$$A_e = \sqrt{2}G_F N_e \equiv A,$$ (20)

$$A_n = \frac{1}{\sqrt{2}}G_F N_n = \frac{1}{2}A\frac{N_n}{N_e}.$$ (21)

However, the interaction terms $U^{-1}V_e U$ and $U^{-1}V_n U$ are complicated[11].

The flavor eigenstate is convenient for describing the interaction. The Hamiltonian in the flavor basis is

$$\begin{aligned}
\mathcal{H}_m^{(\mathrm{flavor})} &= U\mathcal{H}_0 U^{-1} + V_{\mathrm{CC}} + V_{\mathrm{NC}} \\
&= U\mathcal{H}_0 U^{-1} + V_e + V_n - A_n I \\
&= U\mathcal{H}_0 U^{-1} + \mathrm{diag}(A_e, 0, 0, A_n) - A_n I.
\end{aligned}$$ (22)

Although the term of a vacuum Hamiltonian is complicated, the interaction terms are simple for matter densities A_e and A_n.

III.   CALCULATIONS OF RESONANCE CONDITIONS

Some analytical calculations show that the transition probabilities for the neutrino oscillations in matter depend on the matter density[10, 11]. In these analyses, the neutrino

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resonance occurs at the critical (matter) density. The critical density also changes with variation of neutrino energy. The lower neutrino energy reduces, the higher the density to observe the resonance of neutrino becomes.

The resonance of neutrino occurs when the energy levels in the presence of matter approach to each other. To discuss the resonance conditions, we consider the relations between the neutrino energy and the matter density. The neutrino energy is one of eigenvalues of the characteristic equation for the matter Hamiltonian. Therefore, the behavior of the eigenvalues decides the neutrino resonance conditions.

A. Discriminant of the characteristic equation

To simplify discussion, we consider the Hamiltonian $\mathcal{H}_m^{(\text{flavor})}$ in the flavor basis in (22), in which the interaction terms are simple for matter density A. The degree of the characteristic equation of the Hamiltonian is determined by the number of neutrino flavors we suppose, e.g., the characteristic equation for 3-neutrino oscillations is a cubic one. The roots α_i $(i = 1, 2, 3, \cdots)$ of the characteristic equation become more complicated as the degree of the equation increases. It is tedious to solve the characteristic equation of higher degree.

In general, the discriminant $D_{(n)}$ of the characteristic equation

$$x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_0 = 0 \tag{23}$$

with degree n, which has constant coefficients c_k $(k = 0, 1, \cdots, n - 1)$, are expressed by the roots α_i with respect to x of the equation:

$$D_{(n)} = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2. \tag{24}$$

The discriminant $D_{(n)}$ includes the difference between eigenvalues α_i and α_j, that is, a effective energy difference. It suggests that we can use the discriminant $D_{(n)}$ to judge the neutrino resonance conditions. If $D_{(n)}$ becomes small, we can conclude that the resonance occurs, because $D_{(n)}$ depends on the square $(\alpha_i - \alpha_j)^2$ of any energy differences. Therefore, some resonances may occur when one of $(\alpha_i - \alpha_j)^2$ has (local) minimum value.

The discriminant $D_{(n)}$ is polynomials of the roots α_i, of which degree is $2 \times {}_nC_2$, e.g., 6 for the discriminant of the cubic equation. The discriminant $D_{(n)}$ is also written as the symmetric polynomials for α_i. The fundamental theorem of symmetric polynomials tells

us that every symmetric polynomials in α_i $(i = 1, 2, 3, \cdots, n)$ can be represented by the following elementally symmetric polynomials:

$$
\begin{aligned}
\alpha_1 + \alpha_2 + \cdots + \alpha_n &= c_{n-1}, \\
\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{n-1}\alpha_n &= c_{n-2}, \\
\alpha_1\alpha_2\alpha_3 + \cdots + \alpha_{n-2}\alpha_{n-1}\alpha_n &= c_{n-3}, \\
\cdots \quad &, \\
\alpha_1\alpha_2\cdots\alpha_n &= c_0.
\end{aligned}
\tag{25}
$$

The solutions α_i and the coefficients c_k for the characteristic equation are related to each other. Indeed, this relations between α_i and c_k are given by (25). In brief, the discriminant $D_{(n)}$, which includes the effective neutrino energy differences, can be represented by the coefficients of the characteristic equation for the Hamiltonian in matter without solving the characteristic equation.

For example, a discriminant $D_{(2)}$ for a algebraic equation $x^2 + c_0 = 0$ is calculated as follows:

$$
\begin{aligned}
D_{(2)} \equiv (\alpha_1 - \alpha_2)^2 &= (\alpha_1 + \alpha_2)^2 - 4\alpha_1\alpha_2 \\
&= -4c_0.
\end{aligned}
\tag{26}
$$

The concrete form of discriminants $D_{(3)}$ and $D_{(4)}$ are given as follows:

$$
\begin{aligned}
D_{(3)} &= -27c_0{}^2 - 4c_1{}^3 \quad \text{(for } x^3 + c_1 x + c_0 = 0\text{)}, \tag{27} \\
D_{(4)} &= 256c_0{}^3 - 27c_1{}^4 + 144c_0 c_1{}^2 c_2 \\
&\quad -128c_0{}^2 c_2{}^2 - 4c_1{}^2 c_2{}^3 + 16c_0 c_2{}^4 \\
&\quad \text{(for } x^4 + c_2 x^2 + c_1 x + c_0 = 0\text{)}. \tag{28}
\end{aligned}
$$

Note that, for the algebraic equation (23) with degree n, a term $c_{n-1}x^{n-1}$ with power $(n-1)$ of a variable x vanishes by a suitable transformation: $x \to x - c_{n-1}/n$.

B.   Traceless Matrix T_N

In order to find the explicit form of the characteristic equation, the Hamiltonian in the matrix form is separated into the diagonal and the traceless matrices. An arbitrary $N \times N$

matrix M can always be written as

$$M = T_N + \frac{1}{N}(\mathrm{tr}M)I_N, \tag{29}$$

where T_N and I_N are $N \times N$ traceless and unit matrices, respectively. Note that $\mathrm{tr}T_N = 0$.

The characteristic equation for the matrix T_N is written by an eigenvalue λ of T_N as

$$\begin{aligned}
0 &= \det(T_N - \lambda I_N) \\
&= \lambda^N + c_{N-1}\lambda^{N-1} + \cdots + c_2\lambda^2 + c_1\lambda + c_0,
\end{aligned} \tag{30}$$

where the coefficient c_0 is the determinant of T_N, e.g., $c_0 = \det T_N$ and c_k $(k = 1, 2 \cdots N-1)$ are expressed by the sum of cofactors of the diagonal elements of T_N. But $c_{N-1} = 0$, because c_{N-1} is equal to the trace of the traceless matrix T_N.

In the 4-neutrino model, the Hamiltonian in the matrix form is given in (19) and (22). Ignoring the CP violations, the trace of the matrix $\mathcal{H}_m^{(\mathrm{flavor})}$ in (22) is

$$\mathrm{tr}\mathcal{H}_m^{(\mathrm{flavor})} = E_1 + E_2 + E_3 + E_4 + A_e - 3A_n, \tag{31}$$

where we use the unitary conditions, e.g., $U_{e1}^2 + U_{e2}^2 + U_{e3}^2 + U_{e4}^2 = 1$. Then, the Hamiltonian in matter is described as

$$\mathcal{H}_m^{(\mathrm{flavor})} = T_4 + \frac{1}{4}(\mathrm{tr}\mathcal{H}_m^{(\mathrm{flavor})})I_4. \tag{32}$$

The characteristic equation for matrix T is written as

$$\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0. \tag{33}$$

The coefficients c_l $(l = 1, 2, 3, 4)$ are expressed in the matrix form as

$$\begin{aligned}
c_0 &= \det T, \\
c_1 &= -\mathrm{cof}\,T_{(1)} - \mathrm{cof}T_{(2)} - \mathrm{cof}\,T_{(3)} - \mathrm{cof}T_{(4)}, \\
c_2 &= \mathrm{cof}\,T_{(12)} + \mathrm{cof}\,T_{(13)} + \mathrm{cof}\,T_{(14)} \\
&\quad + \mathrm{cof}\,T_{(23)} + \mathrm{cof}\,T_{(24)} + \mathrm{cof}\,T_{(34)}, \\
c_3 &= -\mathrm{tr}T = 0,
\end{aligned}$$

where the cofactors $\mathrm{cof}\,T_{(p)}$ $(p = 1, 2, 3, 4)$ of diagonal components T_{pp} of T_4 and $\mathrm{cof}\,T_{(rs)}$ $(1 \le r < s \le 4)$ of T_{rr} and T_{ss} are determinants of 3×3 and 2×2 matrices, respectively,

e.g. ,

$$\mathrm{cof}\,T_{(2)} \;=\; \sum_{p_1,p_2,p_3,p_4=1}^{4} \epsilon_{p_1 p_2 p_3 p_4} T_{1p_1} \delta_{2p_2} T_{3p_3} T_{4p_4},$$

$$\mathrm{cof}\,T_{(13)} \;=\; \sum_{p_1,p_2,p_3,p_4=1}^{4} \epsilon_{p_1 p_2 p_3 p_4} \delta_{1p_1} T_{2p_2} \delta_{3p_3} T_{4p_4}.$$

The coefficients c_k of the characteristic equation for the matrix T_4 are obtained from the components T_{rs} of T_4, and the discriminant $D_{(4)}$ is derived by these coefficients from (28). The discriminants of T_2 and T_3 in 2- and 3-neutrino models, respectively, are given in the similar method as in 4-neutrino model.

C.   Resonance conditions for 2-neutrino oscillation

The discriminant of the characteristic equation for the matter Hamiltonian is derived by the calculations shown below. For the simple case, a calculation for the discriminant of 2-neutrino Hamiltonian is performed.

The matter Hamiltonian for ν_α and ν_β is described as

$$
\begin{aligned}
\mathcal{H}_m^{(\text{flavor})} \;=\;& U\mathcal{H}_0^{(\text{mass})}U^{-1} + V_{\text{CC}} + V_{\text{NC}} \\
=\;& \begin{bmatrix} E - \frac{E_{21}}{2}\cos 2\theta & \frac{E_{21}}{2}\sin 2\theta \\ \frac{E_{21}}{2}\sin 2\theta & E + \frac{E_{21}}{2}\cos 2\theta \end{bmatrix} \\
& + \begin{bmatrix} A_e \delta_{e\alpha} & 0 \\ 0 & A_n \delta_{s\beta} \end{bmatrix},
\end{aligned}
\tag{34}
$$

where θ is the mixing angle, and

$$E_{21} = E_2 - E_1, \quad E = \frac{1}{2}(E_1 + E_2). \tag{35}$$

The interaction term in (34) includes the factors $A_e \delta_{e\alpha}$ and $A_n \delta_{s\beta}$. These factors appear if we assume the electron neutrino ν_e and the sterile neutrino ν_s, respectively. In the case of 2-neutrino Hamiltonian for ν_e and ν_μ, we can leave $A_n \delta_{s\beta}$ out of consideration.

From (29), the Hamiltonian (34) is rewritten for the 2×2 traceless matrix T_2:

$$\mathcal{H}_m^{(\text{flavor})} \;=\; T_2 + \frac{1}{2}(\mathrm{tr}\mathcal{H}_m^{(\text{flavor})})I_2, \tag{36}$$

$$\mathrm{tr}\mathcal{H}_m^{(\text{flavor})} \;=\; 2E + A_e \delta_{e\alpha} + A_n \delta_{s\beta}. \tag{37}$$

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The concrete form of the traceless matrix T_2 is

$$T_2 = \frac{E_{21}}{2} \begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} + \frac{1}{2}(A_e \delta_{e\alpha} - A_n \delta_{s\beta}) \mathrm{diag}(1, -1). \tag{38}$$

The characteristic equation for the matrix T_2 is expressed as

$$\begin{aligned} 0 &= \det(T_2 - \lambda I_2) \\ &= \lambda^2 - \frac{1}{4}\left[(A_e \delta_{e\alpha} - A_n \delta_{s\beta} - E_{21}\cos 2\theta)^2 + E_{21}{}^2 \sin^2 2\theta\right]. \end{aligned}$$

Then, the discriminant $D_{(2)}$ is presented as follows

$$D_{(2)} = (A_e \delta_{e\alpha} - A_n \delta_{s\beta} - E_{21}\cos 2\theta)^2 + E_{21}^2 \sin^2 2\theta. \tag{39}$$

Therefore, the condition for the neutrino resonance, where (39) gets (local) minimum value, for two neutrinos ν_α and ν_β is

$$A_e \delta_{e\alpha} - A_n \delta_{s\beta} = E_{21}\cos 2\theta. \tag{40}$$

Note that energy differences E_{21} are approximately given by

$$E_{21} = E_2 - E_1 \sim \frac{m_2{}^2 - m_1{}^2}{2E} \equiv \frac{\Delta m_{21}^2}{2E}, \tag{41}$$

where (13) has been used, and Δm_{21}^2 is a squared mass difference.

From (40) and (41), we conclude that, for the 2-neutrino resonance conditions, the average E of the neutrino energies and the factor $(A_e \delta_{e\alpha} - A_n \delta_{s\beta})$ of the matter densities are in inverse proportion to each other:

$$E(A_e \delta_{e\alpha} - A_n \delta_{s\beta}) = \frac{\Delta m_{21}^2}{2}\cos 2\theta. \tag{42}$$

Note that (42) gives a usual expression[16] for the resonance condition if we set $A_n \delta_{s\beta} = 0$.

IV. NUMERICAL ANALYSES OF RESONANCE CONDITIONS

We apply some results obtained in the previous section to the 2-, 3- and 4-neutrino models. A discriminant explicitly contains the neutrino energy differences $E_{ab} = E_a - E_b$ ($a, b = 1, 2, 3, 4$ in the 4-neutrino model) in vacuum. These energy differences are approximately

given by the matter mass differences $\Delta m_{ab}^2 = m_a{}^2 - m_b{}^2$, which are well-known quantities in the various neutrino oscillation experiments. In the similar way as showed in (41),

$$E_{ab} \simeq \frac{\Delta m_{ab}{}^2}{2E}, \tag{43}$$

where E is a average of the neutrino energies, i.e., in four-neutrino model, $E = (E_1 + E_2 + E_3 + E_4)/4$.

These days, three kinds of mass squared differences are known. They are represented by $\Delta m_{\mathrm{solar}}^2$, $\Delta m_{\mathrm{atm}}^2$ and $\Delta m_{\mathrm{LSND}}^2$, which are used as the parameters in the solar and atmospheric oscillations and the LSND experiment. Using these mass squared differences, several distinct types of mass patterns are possible[7]. The phenomenology and the mixing matrix depend on the type of the mass scheme. We concentrate on the discussion on the $(3+1)_1$ - scheme[11], which is one of the several mass patterns in Fig. 1. This mass-scheme includes the usual 3-neutrino mass pattern.

In the mass pattern $(3+1)_1$, there are three close masses and one distinct mass. Let m_4 and Δm_{43}^2 be the distinct mass and the largest mass squared difference, respectively. Three kinds of the neutrino mass squared differences are put as follows [17]:

$$\Delta m_{21}^2 = \Delta m_{\mathrm{solar}}^2 \simeq 7 \times 10^{-5} \, \mathrm{eV}^2, \tag{44}$$

$$\Delta m_{32}^2 = \Delta m_{\mathrm{atm}}^2 \simeq 3 \times 10^{-3} \, \mathrm{eV}^2, \tag{45}$$

$$\Delta m_{41}^2 = \Delta m_{\mathrm{LSND}}^2 \simeq 1 \, \mathrm{eV}^2. \tag{46}$$

Then, the energy differences E_{ab} are expressed by

$$E_{21} \simeq \frac{\Delta m_{21}^2}{2E}, \quad E_{32} \simeq \frac{\Delta m_{32}^2}{2E},$$
$$E_{41} \simeq \frac{\Delta m_{41}^2}{2E}, \quad E_{31} = E_{32} + E_{21},$$
$$E_{42} = E_{41} - E_{21}, \quad E_{43} = E_{41} - E_{31}, \tag{47}$$

where we suppose that the average E of the neutrino energies is $10\,\mathrm{GeV}$ or $10\,\mathrm{MeV}$.

Next, we consider the approximate mixing matrix for the $(3+1)_1$-scheme [18]:

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\cos\epsilon\cos\delta & \frac{1}{\sqrt{2}}\cos\epsilon\cos\delta & \cos\epsilon\sin\delta & \sin\epsilon \\ -\frac{1}{2} - \frac{1}{2}\sin\delta & \frac{1}{2} - \frac{1}{2}\sin\delta & \frac{1}{\sqrt{2}}\cos\delta & 0 \\ \frac{1}{2} - \frac{1}{2}\sin\delta & -\frac{1}{2} - \frac{1}{2}\sin\delta & \frac{1}{\sqrt{2}}\cos\delta & 0 \\ \frac{-1}{\sqrt{2}}\sin\epsilon\cos\delta & \frac{-1}{\sqrt{2}}\sin\epsilon\cos\delta & -\sin\delta\sin\epsilon & \cos\epsilon \end{pmatrix}, \tag{48}$$

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where $\delta = \frac{5}{180}\pi$ and ϵ are small: $0 \leq \epsilon \leq 0.1$. Here the 3×3 sub-matrix that describes the mixing of the three active neutrinos has the bi-maximal form. The mixing matrix of (48) is given from (8), (9) and (10) by taking

$$\theta_{12} = \frac{\pi}{4}, \quad \theta_{23} = \frac{\pi}{4}, \quad \theta_{13} = \delta$$
$$\theta_{14} = \epsilon, \quad \theta_{24} = 0, \quad \theta_{34} = 0,$$
$$\Delta_{13} = \Delta_{14} = \Delta_{24} = 0. \tag{49}$$

In 2- and 3-neutrino models, we can also discuss the resonance condition by a sub-matrix of 4×4 mixing matrix. For example, in order to consider the 3-neutrino model, we may ignore the fourth elements in (48), taking $\epsilon = 0$. In this case, 4×4 mixing matrix (48) is handled as 3×3 matrix and a unity element.

In this setting, we can get a numerical expression of the discriminant $D_{(3)}$ and $D_{(4)}$ as a smooth function of A:

$$
\begin{aligned}
D_{(3)} \sim \frac{1}{E^6} \quad & [6.5 \times 10^{-21} - 8.4 \times 10^{-18}(AE) \\
& +5.2 \times 10^{-12}(AE)^2 - 6.8 \times 10^{-9}(AE)^3 \\
& +2.3 \times 10^{-6}(AE)^4],
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
D_{(4)} \sim \frac{1}{E^{12}} \quad & [4.2 \times 10^{-22} - 1.3 \times 10^{-19}(AE) \\
& +8.1 \times 10^{-14}(AE)^2 - 1.0 \times 10^{-10}(AE)^3 \\
& +3.5 \times 10^{-8}(AE)^4 + 7.1 \times 10^{-8}(AE)^5 \\
& -3.2 \times 10^{-8}(AE)^6 - 1.4 \times 10^{-7}(AE)^7 \\
& -3.4 \times 10^{-8}(AE)^8 + 7.1 \times 10^{-8}(AE)^9 \\
& +3.5 \times 10^{-8}(AE)^{10} \\
& +1.7 \times 10^{-18}(AE)^{11}],
\end{aligned}
\tag{51}
$$

where $\epsilon = 0.1$. The derivative $\frac{dD_{(n)}}{dA}$ of $D_{(n)}$ may be useful to estimate the neutrino resonance conditions. The solutions of the equation $\frac{dD_{(n)}}{dA} = 0$ will give the the turning value of the discriminant $D_{(n)}$. Although, analyses of the derivative of the discriminant $D_{(n)}$ for 3- and 4-neutrino models are complicated, it is expected that the solution of the equation $\frac{dD_{(n)}}{dA} = 0$, that is a resonance condition, is given as an equation for A ,E, and the function of some parameters:

$$AE = f(\Delta m_{ab}^2, \theta_{ab}, \Delta_{ab}). \tag{52}$$

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If the value in the right hand side of the (52) is fixed, e.g, as in (44), (45), (46) and (49), we can conclude that E and A are in inverse proportion to each other.

A. Normalized discriminant

In this analysis, the discriminant of the characteristic equation for the Hamiltonian in matter has a small value. The value of the discriminant in vacuum is given by setting the matter density A to 0. Here we assume that the electron number density N_e is equal to the neutral particle number density N_n : $N_e = N_n$ which means $A_e = A$ and $A_n = \frac{1}{2}A$ in (20) and (21).

The effective energy differences $|\lambda_a - \lambda_b|$ $(a, b = 1, 2, 3, 4, a \neq b)$, which are the differences between eigenvalues λ_a of the Hamiltonian in matter, are functions of the matter density A. If we set $A = 0$, then the effective energies λ_a can be regard as the eigenvalues E_a of the Hamiltonian in vacuum:

$$|\lambda_a - \lambda_b| = |E_a - E_b| \quad \text{for } A = 0. \tag{53}$$

In a 2-neutrino (ν_e, ν_μ) model, from (41), we get a rough estimation for the discriminant $D_{(2)}$:

$$D_{(2)}|_{A=0} = (\lambda_2 - \lambda_1)^2|_{A=0} = (E_2 - E_1)^2 \simeq \left[\frac{\Delta m_{21}^2}{2E}\right]^2$$

$$= \left[\frac{\Delta m_{\text{solar}}^2}{2E}\right]^2 \sim 10^{-29}[\text{eV}^2] \quad \text{(for } E = 10^{10}\text{eV).}$$

$$\tag{54}$$

It may be easy to expect that, in the 3- and 4-neutrino models, the discriminant $D_{(3)}$, $D_{(4)}$ in vacuum have very small value, since the discriminant are defined by the product (24). Indeed, $D_{(3)} \sim 10^{-82}[\text{eV}^6]$ and $D_{(4)} \sim 10^{-142}[\text{eV}^{12}]$ for $A_e = A_n = 0$, e.g., in vacuum, by our rough calculations. Here we set $E = 10\,\text{GeV}$.

In order to concentrate to the argument about the behavior for $D_{(n)}$ as the function of A, we define the normalized discriminant D_n which is expressed as the value of the $D_n(A)$ divided by that for $A = 0$:

$$D_n = \frac{D_{(n)}(A)}{D_{(n)}(A=0)}. \tag{55}$$

14

Hereafter, D_4 means the normalized discriminant for the 4×4 matrix which is the operator of the 4-neutrino $(\nu_e, \nu_\mu, \nu_\tau, \nu_s)$ model. Moreover, D_3 specializes the normalized discriminant for the 3×3 matrix of Hamiltonian, in which we deal with the three active neutrinos: $(\nu_e, \nu_\mu, \nu_\tau)$. In order to compare the 3- and 4-neutrino models with the usual neutrino phenomenology, we consider the two neutrinos (ν_e, ν_μ) for D_2.

B. Graphical expression of resonance conditions

As an illustration of the resonance phenomena, the normalized discriminants D_n ($n = 2, 3, 4$) are plotted as a function of the matter density A in Fig. 2. Here we set the average E of the neutrino energies to $10\,\text{GeV}$, which is a value of neutrino energy used in previous analyses [10, 11].

From Fig. 2, the following results can be derived. If the mixing angles θ_{13} and θ_{14} are small, the third- and fourth-neutrino effects appear as a resonance. The sharp drop in the Fig. 2 is a good reflection of the neutrino resonance. Indeed, the surviving probabilities P_{ee}, which are discussed in [10] and [11], for neutrino oscillations in matter have sharp drops. The smaller a mixing angle reduces, the sharper the drops for the discriminant $D_{(n)}$ and the surviving probabilities become.

One effect for the third neutrino ν_τ appears beyond $A \sim 10^{-13}\,\text{eV}$. The other effect for the fourth neutrino ν_s also occurs beyond $A \sim 10^{-10}\,\text{eV}$. It means that the matter density to cause the resonance for $\nu_e \leftrightarrow \nu_s$ or $\nu_e \leftrightarrow \nu_\tau$ varies with the squared mass differences:

$$\frac{10^{-10}}{10^{-13}} = 10^3 \sim \frac{\Delta m^2_{\text{LSND}}}{\Delta m^2_{\text{atm}}}. \tag{56}$$

The above results are argued for $E = 10\,\text{GeV}$. The normalized discriminant for $E = 10\,\text{MeV}$ is shown in Fig. 3. The lines in Fig. 2 and Fig. 3 have the similar shapes except for the value of the matter density A. The difference of two results occurs due to the difference of the average E of neutrino energies. For instance, one resonance for $\nu_e \leftrightarrow \nu_\tau$ in Fig. 2 ($E = 10\,\text{GeV}$) occurs for $A \sim 10^{-13}\,\text{eV}$, the other resonance for $\nu_e \leftrightarrow \nu_\tau$ in Fig. 3 ($E = 10\,\text{MeV}$) occurs at $A \sim 10^{-10}\,\text{eV}$. Therefore the average E of the neutrino energies and the matter density A are in inverse proportion to each other as shown in (42), i.e.,

$$EA = \frac{\Delta m^2_{\text{atm}}}{2} \cos 2\theta_{13} \sim 10^{-3} [\text{eV}^2]. \tag{57}$$

Note that we set the matter density to $A_e = A$, $A_n = (1/2)A$. In the case of the resonance $\nu_e \leftrightarrow \nu_s$,

$$EA = \Delta m_{\mathrm{LSND}}^2 \cos 2\theta_{14} \sim 1[\mathrm{eV}^2].\tag{58}$$

V. DISCUSSION

The main result of our analysis is derivation of the discriminant of the characteristic equation for the Hamiltonian in matter, which is discussed in flavor basis. The concrete form of the discriminant for 2-, 3-, and 4-neutrino models is given in (26), (27), and (28), respectively. The effective Hamiltonian in the 2-neutrino model is also calculated concretely. The discriminant made from the effective Hamiltonian is a function of the matter density A and the average E of the neutrino energies. The calculations for 3- and 4-neutrino models are straightforward but the forms of the discriminants are complicated. One of the numerical expression of the discriminant is given in (50) and (51). The graphical expression of the discriminant are also shown in Fig.2 and Fig.3.

One of the resonance conditions for the neutrino oscillation is described in (42). It shows that E and A are in inverse proportion to each other, as confirmed in Fig. 2 and Fig. 3. That is, the lower the neutrino energy reduces the higher the density to observe the resonance between the neutrinos becomes. The concrete form of the resonance condition for 2-neutrino model is shown in (57) and (58). From (50) and (51), the discriminant $D_{(n)}$ is a smooth function of A. The derivative $\frac{dD_{(n)}}{dA}$ of $D_{(n)}$ may be useful to estimate the neutrino resonance conditions. Although, analyses of the derivative of the discriminant $D_{(n)}$ for 3- and 4-neutrino models are complicated, we can expect that the neutrino resonance condition will be given as (52).

The conditions that the discriminant $D_{(n)}$ has a local minimum value give the resonance conditions for neutrino oscillation. In our analysis, it is expected that the resonance of $\nu_e \leftrightarrow \nu_\tau$ and $\nu_e \leftrightarrow \nu_s$ occur at $A = 10^{-13}\,\mathrm{eV}$ and $A = 10^{-10}\,\mathrm{eV}$, respectively, for $E = 10\,\mathrm{GeV}$. This results can be confirmed by the previous calculations [10, 11] in 2-, 3- and 4-neutrino models, respectively. For instance, the surviving probabilities P_{ee} for 4-neutrino oscillation is given as

$$P_{ee} = 1 - 4\sum_{a=1}^{3}\sum_{b=a+1}^{4} B_a B_b \sin^2\left[\frac{L}{2}\cdot\frac{\Delta m_{ab}^2}{2E}\right],\tag{59}$$

where B_a is a factor determined by the mixing matrix and the characteristic equation of the

matter Hamiltonian in mass basis[11]. The factor L is a baseline length which stands for the neutrino flying distance. Fig. 4 is a result of the recalculation of the transition probabilities for new conditions which we have used in this paper. One can find that the value of the matter density A that the resonance occurs in Fig. 2 coincides with the value of A that the sharp drop of the transition probabilities occurs in Fig.4.

Is the condition to observe the neutrino resonance realistic? The electron matter density A is 1×10^{-11} eV at the center of the sun. The average of the electron matter density in earth is about 5×10^{-13} eV [11]. In the case of $E = 10\,\text{GeV}$, it may be possible to observe the resonance between ν_e and ν_τ since the condition is $A \sim 10^{-13}$ eV which corresponds to the density of the earth. But in other analysis it may be difficult to observe the resonance.

One may consider that the resonance will occur in the high energy and low matter density, which is restricted by (42). But there is a limit of the neutrino energy to observe the neutrino oscillation. From (59), the baseline length L to observe the neutrino oscillation and the average E of the neutrino energies are proportional to each other. For instance, the condition to observe a neutrino transition $\nu_e \rightarrow \nu_s$ for $E = 10\,\text{GeV}$ and $E = 10\,\text{MeV}$ is $L > 25000\,\text{m}$ and $L > 25\,\text{m}$, respectively, where we assume the condition

$$\frac{L}{\Lambda/4} = \frac{\Delta m_{ab}{}^2 \cdot L}{2\pi E} > 1. \tag{60}$$

Then, we have considered that the neutrino oscillation can be observed when the neutrino flying distance is more than $\Lambda/4$, which Λ corresponds to the wavelength in (59).

The 3-neutrino model with a small and bi-maximal mixings and the 4-neutrino model will be tested more closely by the upcoming experiments. In this paper, the resonance conditions for 2-, 3-, and 4-neutrino mixings are given without a preconception. Our result of the resonance conditions, which are given by using the discriminant, does not deny the possibility of the 4-neutrino oscillation, but the observation of the resonance is not realistic.

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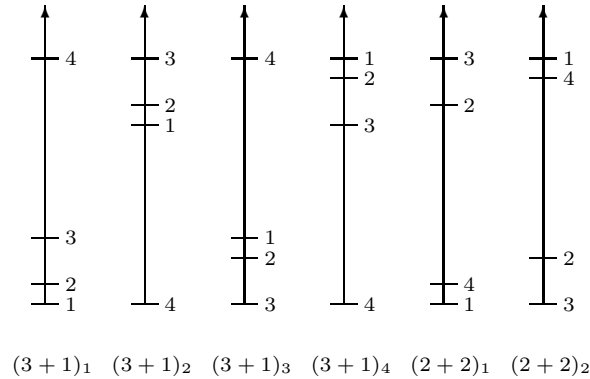[18] V. Barger, B. Kayser, J. Learned, T. Weiler and K. Whisnant, Phys. Lett. B **489**, 345 (2000).
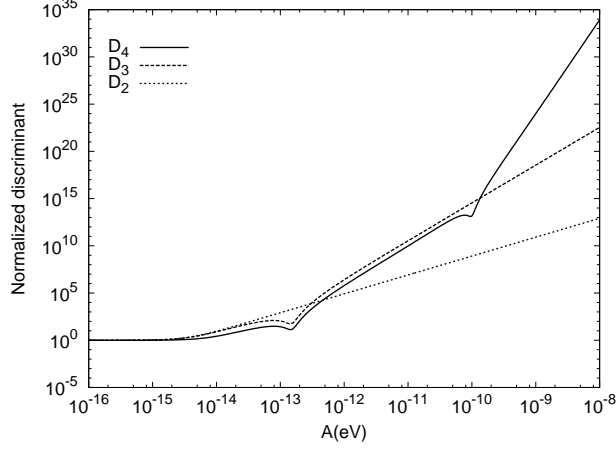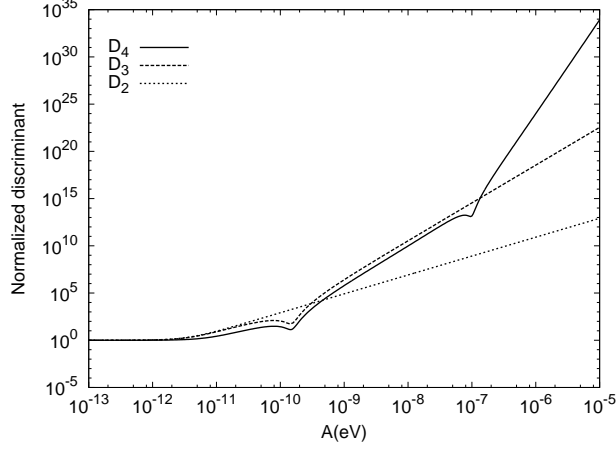
FIG. 1: Several mass patterns for 4-neutrino schemes.

FIG. 2: Normalized discriminants D_n $(n = 2, 3, 4)$ as a function of the matter density A for a $(3 + 1)_1$-scheme, where $\theta_{12} = \frac{\pi}{4}$, $\theta_{23} = \frac{\pi}{4}$, $\theta_{13} = \frac{5}{180}\pi$, $\theta_{14} = \frac{3}{180}\pi$, $\theta_{24} = 0$, $\theta_{34} = 0$, $\Delta_{13} = \Delta_{24} = \Delta_{34} = 0$, $\Delta m_{21}^2 \simeq 7 \times 10^{-5}\,\mathrm{eV}^2$, $\Delta m_{32}^2 \simeq 3 \times 10^{-3}\,\mathrm{eV}^2$, $\Delta m_{41}^2 \simeq 1\,\mathrm{eV}^2$, $E = 10$ GeV. The solid, broken and dotted lines are normalized discriminants D_4, D_3 and D_2, respectively.

FIG. 3: Normalized discriminants D_n ($n = 2, 3, 4$) as a function of the matter density A for a $(3+1)_1$-scheme, where $\theta_{12} = \frac{\pi}{4}$, $\theta_{23} = \frac{\pi}{4}$, $\theta_{13} = \frac{5}{180}\pi$, $\theta_{14} = \frac{3}{180}\pi$, $\theta_{24} = 0$, $\theta_{34} = 0$, $\Delta_{13} = \Delta_{24} = \Delta_{34} = 0$, $\Delta m_{21}^2 \simeq 7 \times 10^{-5}\,\text{eV}^2$, $\Delta m_{32}^2 \simeq 3 \times 10^{-3}\,\text{eV}^2$, $\Delta m_{41}^2 \simeq 1\,\text{eV}^2$, $E = 10$ MeV. The solid, broken and dotted lines are normalized discriminants D_4, D_3 and D_2, respectively.
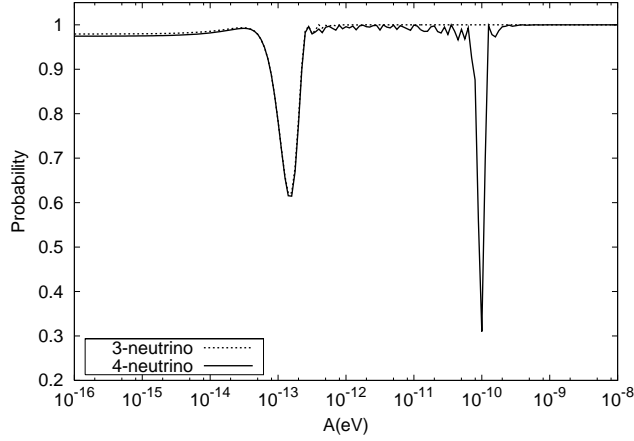

FIG. 4: The surviving probability P_{ee} of the electron neutrino transition as a function of the matter density A for the $(3+1)_1$-scheme, where $\theta_{12} = \frac{\pi}{4}$, $\theta_{23} = \frac{\pi}{4}$, $\theta_{13} = \frac{5}{180}\pi$, $\theta_{14} = \epsilon$, $\theta_{24} = 0$, $\theta_{34} = 0$, $\Delta_{13} = \Delta_{24} = \Delta_{34} = 0$, $\Delta m_{21}^2 \simeq 7 \times 10^{-5}\,\text{eV}^2$, $\Delta m_{32}^2 \simeq 3 \times 10^{-3}\,\text{eV}^2$, $\Delta m_{41}^2 \simeq 1\,\text{eV}^2$, $E = 10\,\text{GeV}$ and $L = 10000\,\text{km}$. The solid and broken lines show the transition probabilities for $\epsilon = \frac{3}{180}\pi, 0$, respectively, which correspond to the 4-neutrino and the 3-neutrino cases.